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Representations of the q-deformed algebra $U_q(\mathrm{iso}_2)$

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Abstract. An algebra homomorphism ψ from the q-deformed algebra $U_q(\mathrm{iso}_2)$ with generating elements I, T_1, T_2 and defining relations $[I, T_2]_q = T_1$, $[T_1, I]_q = T_2$, $[T_2, T_1]_q = 0$ (where $[A, B]_q = q^{1/2}AB - q^{-1/2}BA$) to the extension $\hat{U}_q(\mathrm{m}_2)$ of the Hopf algebra $U_q(\mathrm{m}_2)$ is constructed. The algebra $U_q(\mathrm{iso}_2)$ at $q = 1$ leads to the Lie algebra $\mathrm{iso}_2 \sim \mathrm{m}_2$ of the group $ISO(2)$ of motions of the Euclidean plane. The Hopf algebra $U_q(\mathrm{m}_2)$ (which is not isomorphic to $U_q(\mathrm{iso}_2)$) is treated as a Hopf q-deformation of the universal enveloping algebra of iso_2 and is well known in the literature.

Not all irreducible representations of $U_q(\mathrm{m}_2)$ can be extended to representations of the extension $\hat{U}_q(\mathrm{m}_2)$. Composing the homomorphism ψ with irreducible representations of $\hat{U}_q(\mathrm{m}_2)$ we obtain representations of $U_q(\mathrm{iso}_2)$. Not all of these representations of $U_q(\mathrm{iso}_2)$ are irreducible. The reducible representations of $U_q(\mathrm{iso}_2)$ are decomposed into irreducible components. In this way we obtain all irreducible representations of $U_q(\mathrm{iso}_2)$ when q is not a root of unity. A part of these representations turns into irreducible representations of the Lie algebra iso_2 when $q \to 1$. Representations of the other part have no classical analogue.

1. Introduction

Soon after the definition of Drinfeld–Jimbo algebras $U_q(g)$, corresponding to semisimple Lie algebras g, the Hopf algebra $U_q(\mathrm{m}_2)$ was defined [1] which is treated as a q-deformation of the universal enveloping algebra of the Lie algebra iso_2 of the group of motions of the Euclidean plane (for the description of this group, its Lie algebra and their representations see, for example, [2], chapter 4).

However, there is another q-deformation of the universal enveloping algebra $U(\mathrm{iso}_2)$ of the Lie algebra iso_2 which will be denoted by $U_q(\mathrm{iso}_2)$. In the general form (that is, for $U(\mathrm{iso}_n)$) such q-deformations were defined in [3]. The Hopf algebra $U_q(\mathrm{m}_2)$ is related to the well known quantum algebra $U_q(\mathrm{sl}_2)$, while the associative algebra $U_q(\mathrm{iso}_2)$ is connected with the non-standard q-deformation $U_q(\mathrm{so}_3)$ of the universal enveloping algebra $U(\mathrm{so}_3)$ which is sometimes called the Fairlie algebra.

It is known that the theory of representations of the associative algebra $U_q(\mathrm{so}_3)$ is richer than that of the algebra $U_q(\mathrm{sl}_2)$ [4–6]. It was shown recently [7] that the theory of representations of the algebra $U_q(\mathrm{iso}_2)$ is also richer than that of the algebra $U_q(\mathrm{m}_2)$. In particular, the algebras $U_q(\mathrm{so}_3)$ and $U_q(\mathrm{iso}_2)$ have irreducible representations of non-classical type (that is, representations which have no limit at $q \to 1$). The paper [7] is devoted to study of irreducible $*$-representations of the algebra $U_q(\mathrm{iso}_2)$ equipped with $*$-structures. Irreducible representations of $U_q(\mathrm{iso}_2)$ of the classical type are given in [8].

The aim of the present paper is to study irreducible representations of $U_q(\mathrm{iso}_2)$ when this algebra is not equipped with some $*$-structure and to clarify why irreducible representations

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of $U_q(\mathrm{iso}_2)$ of the non-classical type appear. We do this in the same way as in the case of representations of the algebra $U_q(\mathrm{so}_3)$ in [6]. Namely, we relate the algebra $U_q(\mathrm{iso}_2)$ with the extension $\hat{U}_q(\mathrm{m}_2)$ of the Hopf algebra $U_q(\mathrm{m}_2)$. This allows us to obtain representations of $U_q(\mathrm{iso}_2)$ from those of the extended algebra $\hat{U}_q(\mathrm{m}_2)$. We prove that if q is not a root of unity, then irreducible representations obtained in this way exhaust, up to equivalence, all irreducible representations of $U_q(\mathrm{iso}_2)$.

Note that representations of $U_q(\mathrm{iso}_2)$ are more complicated than these of $U_q(\mathrm{m}_2)$ since the first ones have no lowering and raising operators. For this reason, the representation theory for $U_q(\mathrm{iso}_2)$ is more complicated than that for $U_q(\mathrm{m}_2)$.

2. The algebras $U_q(\mathrm{iso}_2)$ and $\hat{U}_q(\mathrm{m}_2)$

The algebra $U_q(\mathrm{iso}_2)$ is obtained by a q-deformation of the standard commutation relations

$$[I, T_2] = T_1 \qquad [T_1, I] = T_2 \qquad [T_2, T_1] = 0$$

of the Lie algebra iso_2. So, $U_q(\mathrm{iso}_2)$ is defined [7, 8] as the complex associative algebra with unit element generated by the elements I, T_1, T_2 satisfying the defining relations

$$[I, T_2]_q := q^{1/2} I T_2 - q^{-1/2} T_2 I = T_1 \tag{1}$$

$$[T_1, I]_q := q^{1/2} T_1 I - q^{-1/2} I T_1 = T_2 \tag{2}$$

$$[T_2, T_1]_q := q^{1/2} T_2 T_1 - q^{-1/2} T_1 T_2 = 0. \tag{3}$$

Note that the elements T_2 and T_1 of the algebra $U_q(\mathrm{iso}_2)$ do not commute (as it is a case in the algebra iso_2; these elements correspond to shifts along the axes of the plane). We say that they q-commute, that is, $q^{1/2} T_2 T_1 - q^{-1/2} T_1 T_2 = 0$. This means that they generate the associative algebra determining the quantum plane.

Unfortunately, a Hopf algebra structure is not known on $U_q(\mathrm{iso}_2)$. However, it can be embedded into the Hopf algebra $U_q(\mathrm{isl}_2)$ as a Hopf co-ideal. (The algebra $U_q(\mathrm{isl}_2)$ is the q-deformation of the universal enveloping algebra $U(\mathrm{isl}_2)$ of the Lie algebra isl_2 of the inhomogeneous Lie group $ISL(2)$).

Let us remark that the Hopf algebra $U_q(\mathrm{m}_2)$ is a Hopf dual for the algebra of functions on the quantum Euclidean group $\mathcal{F}(ISO_q(2))$. The algebras of the type $U_q(\mathrm{iso}_2)$ appear in quantum gravity [9].

The relations (1)–(3) lead to the Poincaré–Birkhoff–Witt theorem for the algebra $U_q(\mathrm{iso}_2)$ which can be formulated as: *the elements* $T_1^j T_2^k I^l$, $j, k, l = 0, 1, 2, \ldots,$ *form a basis of the linear space* $U_q(\mathrm{iso}_2)$.

Indeed, by using the relations (1)–(3) any product of the elements I, T_2, T_1 can be reduced to a sum of the elements $T_1^j T_2^k I^l$ with complex coefficients. Using the diamond lemma [10] (or its special case from subsection 4.1.5 in [11]) it is proved that these elements are linearly independent.

Note that by (1) the element T_1 is not independent: it is determined by the elements I and T_2. Thus, the algebra $U_q(\mathrm{iso}_2)$ is generated by I and T_2, but now instead of quadratic relations (1)–(3) we must take the relations

$$I^2 T_2 - (q + q^{-1}) I T_2 I + T_2 I^2 = -T_2 \tag{4}$$

$$I T_2^2 - (q + q^{-1}) T_2 I T_2 + T_2^2 I = 0 \tag{5}$$

which are obtained if we substitute expression (1) for T_1 into (2) and (3). The equation $q^{1/2} I T_2 - q^{-1/2} T_2 I = T_1$ and the relations (4) and (5) restore the relations (1)–(3).

Note that relation (5) is a relation of Serre's type in the definition of quantum algebras by Drinfeld and Jimbo. Relation (4) differs from Serre's relation by the appearance of a non-vanishing right-hand side.

It is known that the element $C = T_1^2 + T_2^2$ from the universal enveloping algebra $U(\mathrm{iso}_2)$ belongs to the centre of this algebra. The analogue of this element in $U_q(\mathrm{iso}_2)$ is the element $C_q = \frac{1}{2}(T_1T_1' + T_1'T_1) + \frac{1}{2}(q + q^{-1})T_2^2$, where $T_1' = q^{-1/2}IT_2 - q^{1/2}T_2I$ (see [8]), that is $[C_q, X] := C_qX - XC_q = 0$ for all $X \in U_q(\mathrm{iso}_2)$. This element can be reduced to the form

$$C_q = q^{-1}T_1^2 + qT_2^2 + q^{-3/2}(1 - q^2)T_1T_2I. \tag{6}$$

The algebra $U_q(\mathrm{iso}_2)$ is closely related to (but does not coincide with) the quantum algebra $U_q(\mathrm{m}_2)$. The last algebra is generated by the elements q^H, q^{-H}, E, F satisfying the relations

$$\begin{aligned} &q^Hq^{-H} = q^{-H}q^H = 1 \\ &q^HEq^{-H} = qE \\ &q^HFq^{-H} = q^{-1}F \\ &[E, F] := EF - FE = 0. \end{aligned} \tag{7}$$

In order to relate the algebras $U_q(\mathrm{iso}_2)$ and $U_q(\mathrm{m}_2)$ we need to extend $U_q(\mathrm{m}_2)$ by the elements $(q^kq^H + q^{-k}q^{-H})^{-1}$, $k \in \mathbb{Z}$, in the sense of [12]. This extension $\hat{U}_q(\mathrm{m}_2)$ is defined as the associative algebra (with unit element) generated by the elements

$$q^H \qquad q^{-H} \qquad E \qquad F \qquad (q^kq^H + q^{-k}q^{-H})^{-1} \qquad k \in \mathbb{Z}$$

satisfying the defining relations (7) of the algebra $U_q(\mathrm{m}_2)$ and the following natural relations:

$$(q^kq^H + q^{-k}q^{-H})^{-1}(q^kq^H + q^{-k}q^{-H}) = (q^kq^H + q^{-k}q^{-H})(q^kq^H + q^{-k}q^{-H})^{-1} = 1 \tag{8}$$

$$q^{\pm H}(q^kq^H + q^{-k}q^{-H})^{-1} = (q^kq^H + q^{-k}q^{-H})^{-1}q^{\pm H} \tag{9}$$

$$(q^kq^H + q^{-k}q^{-H})^{-1}E = E(q^{k+1}q^H + q^{-k-1}q^{-H})^{-1} \tag{10}$$

$$(q^kq^H + q^{-k}q^{-H})^{-1}F = F(q^{k-1}q^H + q^{-k+1}q^{-H})^{-1}. \tag{11}$$

3. The algebra homomorphism $U_q(\mathrm{iso}_2) \to \hat{U}_q(\mathrm{m}_2)$

The aim of this section is to give (in an explicit form) the homomorphism of the algebra $U_q(\mathrm{iso}_2)$ to $\hat{U}_q(\mathrm{m}_2)$.

Proposition 1. *There exists a unique algebra homomorphism* $\psi : U_q(\mathrm{iso}_2) \to \hat{U}_q(\mathrm{m}_2)$ *such that*

$$\psi(I) = \frac{\mathrm{i}}{q - q^{-1}}(q^H - q^{-H}) \tag{12}$$

$$\psi(T_2) = (E - F)(q^H + q^{-H})^{-1} \tag{13}$$

$$\psi(T_1) = (\mathrm{i}q^{H-1/2}E + \mathrm{i}q^{-H-1/2}F)(q^H + q^{-H})^{-1} \tag{14}$$

where $q^{H+a} := q^Hq^a$ *for* $a \in \mathbb{C}$.

Proof. In order to prove this proposition we have to show that the defining relations

$$\begin{aligned} &q^{1/2}\psi(I)\,\psi(T_2) - q^{-1/2}\psi(T_2)\,\psi(I) = \psi(T_1) \\ &q^{1/2}\psi(T_1)\,\psi(I) - q^{-1/2}\psi(I)\,\psi(T_1) = \psi(T_2) \\ &q^{1/2}\psi(T_2)\,\psi(T_1) - q^{-1/2}\psi(T_1)\,\psi(T_2) = 0 \end{aligned} \tag{15}$$

of $U_q(\mathrm{iso}_2)$ are satisfied. Let us prove the relation (15). (Other relations are proved similarly.) Substituting the expressions (12)–(14) for $\psi(I)$, $\psi(T_2)$, $\psi(T_1)$ into (15) we obtain (after multiplying both sides of equality by $(q^H + q^{-H})$ on the right) the relation

$$\begin{aligned} q(E-F)Eq^H(qq^H+q^{-1}q^{-H})^{-1} &+ q(E-F)Fq^{-H}(q^{-1}q^H+qq^{-H})^{-1} \\ &-qE^2q^H(qq^H+q^{-1}q^{-H})^{-1} - q^{-1}FEq^{-H}(qq^H+q^{-1}q^{-H})^{-1} \\ &+q^{-1}EFq^H(q^{-1}q^H+qq^{-H})^{-1} + qF^2q^{-H}(q^{-1}q^H+qq^{-H})^{-1} = 0. \end{aligned}$$

Equation (15) is true if and only if this relation is correct. We multiply both its sides by $(qq^H + q^{-1}q^{-H})(q^{-1}q^H + qq^{-H})$ on the right and obtain a relation which does not depend on the generators $(q^kq^H + q^{-k}q^{-H})^{-1}$. This relation is easily verified by using the defining relations (7) of the algebra $U_q(\mathrm{m}_2)$. The proposition is proved. □

Note that $U_q(\mathrm{m}_2)$ is a Hopf algebra. However, it is not possible to induce a Hopf algebra structure on $U_q(\mathrm{iso}_2)$ by means of the homomorphism ψ since an application of the comultiplication Δ of $U_q(\mathrm{m}_2)$ to the elements on the right-hand sides of (12)–(14) gives elements which cannot be expressed as polynomials of $\psi(I)$, $\psi(T_2)$, $\psi(T_1)$.

4. Definition of representations of $U_q(\mathrm{m}_2)$ and $U_q(\mathrm{iso}_2)$

From this point we assume that q is not a root of unity. Let us define representations of the algebras $U_q(\mathrm{m}_2)$ and $U_q(\mathrm{iso}_2)$.

Definition. *By a representation π of $U_q(\mathrm{m}_2)$ (respectively, $U_q(\mathrm{iso}_2)$) we mean a homomorphism of $U_q(\mathrm{sl}_2)$ (respectively, $U_q(\mathrm{iso}_2)$) into the algebra of linear operators (bounded or unbounded) on a Hilbert space $\mathcal{H}$, defined on an everywhere dense invariant subspace $\mathcal{D}$, such that the operator $\pi(q^H)$ (respectively, the operator $\pi(I)$) can be diagonalized, has a discrete spectrum (with finite multiplicities of spectral points if π is irreducible) and its eigenvectors belong to $\mathcal{D}$. Two representations π and π' of $U_q(\mathrm{m}_2)$ (of $U_q(\mathrm{iso}_2)$) on spaces $\mathcal{H}$ and $\mathcal{H}'$, respectively, are called (algebraically) equivalent if there exist everywhere dense invariant subspaces $V \subset \mathcal{H}$ and $V' \subset \mathcal{H}'$ and a one-to-one linear operator $A : V \to V'$ such that $A\pi(a)v = \pi'(a)Av$ for all $a \in U_q(\mathrm{m}_2)$ (respectively, for all $a \in U_q(\mathrm{iso}_2)$) and $v \in V$.*

Remark. Note that the element $I \in U_q(\mathrm{iso}_2)$ corresponds to the homogeneous part of the motion group *ISO*(2). As in the classical case, it is natural to demand in the definition of representations of $U_q(\mathrm{iso}_2)$ that the operator $\pi(I)$ has a discrete spectrum (with finite multiplicities of spectral points for irreducible representations π). Such representations correspond to Harish–Chandra modules of Lie algebras. Note that irreducible $*$-representations of $U_q(\mathrm{iso}_2)$ without a requirement that $\pi(I)$ has a discrete spectrum were studied in [7]. It was shown there that the classification of irreducible $*$-representations by self-adjoint operators in this case is equivalent to the classification of arbitrary families of bounded self-adjoint operators. The classification of irreducible representations (not obligatory $*$-representations) in this case turn into an unsolved problem.

The algebra $U_q(\mathrm{m}_2)$ has the following non-trivial irreducible representations:

(a) one-dimensional representations π_σ, $\sigma \in \mathbb{C}$, $\sigma \neq 0$, determined by the formulae $\pi_\sigma(q^H) = \sigma, \pi_\sigma(E) = \pi_\sigma(F) = 0$;

(b) infinite-dimensional representations π_{rs}, $r, s \in \mathbb{C}$, $r, s \neq 0$, acting on the Hilbert space $\mathcal{H}$ with a basis $|m\rangle$, $m \in \mathbb{Z}$, by the formulae

$$\begin{aligned} &\pi_{rs}(q^H)|m\rangle = sq^m|m\rangle \qquad &&\pi_{rs}(E)|m\rangle = r|m+1\rangle \\ &\pi_{rs}(F)|m\rangle = r|m-1\rangle \qquad &&m \in \mathbb{Z}. \end{aligned} \tag{16}$$

We take $\mathcal{D} = \mathrm{lin}\{|m\rangle | m \in \mathbb{Z}\}$. A direct verification shows that the representations π_{rs} and $\pi_{r's'}$ $(r, s, r', s' \in \mathbb{C}\backslash\{0\})$ are equivalent if and only if $r = \pm r'$ and $s' = q^n s$ for some $n \in \mathbb{Z}$.

Repeating the reasoning of section 5.2 from the book [11] we easily prove that *every irreducible representation of $U_q(\mathrm{m}_2)$ is equivalent to one of the representations* (16) *or is one dimensional.*

Note that at first glance it seems that there exist infinite-dimensional irreducible representations of $U_q(\mathrm{m}_2)$ not equivalent to the representations (16). For example, the formulae $\pi(q^H)|m\rangle = sq^m|m\rangle$, $\pi(E)|m\rangle = r|m+1\rangle$, $\pi(F)|m\rangle = t|m-1\rangle$ also give a representation of $U_q(\mathrm{m}_2)$. However, we make a rescaling of the basis elements by setting $|m\rangle = a^m|m\rangle'$ and obtain $\pi(E)|m\rangle' = ar|m+1\rangle'$ and $\pi(F)|m\rangle' = a^{-1}t|m-1\rangle'$. Setting $a^2 = t/r$, we have the representation π in the form (16) with r replaced by $\sqrt{rt}$.

Note that for $q \to 1$ the representations π_{rs} of $U_q(\mathrm{m}_2)$ turn into irreducible representations of the universal enveloping algebra $U(\mathrm{m}_2)$, that is, all irreducible representations of $U_q(\mathrm{m}_2)$ are deformations of the corresponding irreducible representations of $U(\mathrm{m}_2)$.

We try to extend representations $\pi_{r,s}$ of $U_q(\mathrm{m}_2)$ to representations of the extension $\hat{U}_q(\mathrm{m}_2)$ by using the relation

$$\pi((q^k q^H + q^{-k} q^{-H})^{-1}) := (q^k \pi(q^H) + q^{-k}\pi(q^{-H}))^{-1} \qquad k \in \mathbb{Z}. \tag{17}$$

Clearly, only those irreducible representations $\pi_{r,s}$ of $U_q(\mathrm{m}_2)$ can be extended to $\hat{U}_q(\mathrm{m}_2)$ for which the operators $q^k\pi(q^H) + q^{-k}\pi(q^{-H})$ are invertible. From formulae (16) it is clear that these operators are always invertible for the representations π_{rs}, $s \neq \pm\mathrm{i}q^n$, $n \in \mathbb{Z}$. (For the representations π_{rs}, $s = \pm\mathrm{i}q^n$ for some $n \in \mathbb{Z}$, some of these operators are not invertible since they have zero eigenvalue.) Denoting the extended representations by the same symbols π_{rs}, we can formulate the following statement:

> The algebra $\hat{U}_q(\mathrm{m}_2)$ has the infinite-dimensional representations π_{rs}, $r, s \in \mathbb{C}\backslash\{0\}$, $s \neq \pm\mathrm{i}q^n$ for all $n \in \mathbb{Z}$, given by the relations (16) and (17). The representations π_{rs} and $\pi_{r',s'}$ $(r, s, r', s' \in \mathbb{C}\backslash\{0\}$, $s, s' \neq \pm\mathrm{i}q^n$ for all $n \in \mathbb{Z})$ are equivalent if and only if $r = \pm r'$ and $s' = q^m s$ for some $m \in \mathbb{Z}$. Any irreducible representation of $\hat{U}_q(\mathrm{sl}_2)$ is equivalent to the representation $\pi_{r,s}$ for some r, s or is a one-dimensional representation.

5. Irreducible representations of $U_q(\mathrm{iso}_2)$

If π is a representation of the algebra $\hat{U}_q(\mathrm{m}_2)$ on a space $\mathcal{H}$, then the mapping $R : U_q(\mathrm{iso}_2) \to \mathcal{H}$ defined as the composition $R = \pi \circ \psi$, where ψ is the homomorphism from proposition 1, is a (not necessary irreducible) representation of $U_q(\mathrm{iso}_2)$.

Let us consider the representations

$$R_{rs} = \pi_{rs} \circ \psi$$

of $U_q(\mathrm{iso}_2)$, where π_{rs} are the irreducible representations of $\hat{U}_q(\mathrm{m}_2)$ from the previous section. Using formulae (16) and (12)–(14) we find that

$$R_{rs}(I)|m\rangle = \mathrm{i}\frac{sq^m - s^{-1}q^{-m}}{q-q^{-1}}|m\rangle \tag{18}$$

$$R_{rs}(T_2)|m\rangle = \frac{r}{sq^m + s^{-1}q^{-m}}\{|m+1\rangle - |m-1\rangle\} \tag{19}$$

$$R_{rs}(T_1)|m\rangle = \frac{\mathrm{i}q^{1/2}r}{sq^m + s^{-1}q^{-m}}\{sq^m|m+1\rangle + q^{-m}s^{-1}|m-1\rangle\}. \tag{20}$$

We consider that these operators are defined on the invariant subspace $\mathcal{D} \subset \mathcal{H}$ which is the span of the basis vectors $|m\rangle$. Thus we proved the following assertion: *let* $r, s \in \mathbb{C}\backslash\{0\}$, $s \neq \pm\mathrm{i}q^n$ *for all* $n \in \mathbb{Z}$. *Then formulae (18)–(20) give a representation* R_{rs} *of the algebra* $U_q(\mathrm{iso}_2)$.

The equivalence relations for the above representations of the algebra $\hat{U}_q(\mathrm{m}_2)$ lead to the corresponding equivalence relations for the representations R_{rs}.

Proposition 2. *The representations* R_{rs} *of* $U_q(\mathrm{iso}_2)$ *are irreducible if* $s \neq \pm\,\mathrm{i}q^{m+1/2}$, $m \in \mathbb{Z}$.

Proof. Let $\{a\} := (sq^a - s^{-1}q^{-a})/(q-q^{-1})$. To prove this proposition we first note that since q is not a root of unity and $s \neq \pm\mathrm{i}q^{m+1/2}$, $m \in \mathbb{Z}$, the eigenvalues $\mathrm{i}\{m\}$, $m = 0, \pm 1, \pm 2, \ldots,$ of the operator $R_{rs}(I)$ are pairwise different.

Let $V \subset \mathcal{D}$ be an invariant subspace of the representation R_{rs}. We need to show that $V = \mathcal{D}$. Let $v = \sum_{m_i} \alpha_i|m_i\rangle \in V$, where $|m_i\rangle$ are eigenvectors of $R_{rs}(I)$ which are basis vectors of $\mathcal{H}$. (Note that the sum is finite since $v \in \mathcal{D}$.) Let us prove that $|m_i\rangle \in V$. We prove this for the case when $v = \alpha_1|m_1\rangle + \alpha_2|m_2\rangle$. (The case of a greater number of summands is proved similarly.) We have

$$v' := R_{rs}(I)v = \mathrm{i}\alpha_1\{m_1\}|m_1\rangle + \mathrm{i}\alpha_2\{m_2\}|m_2\rangle.$$

Since $v, v' \in V$, one derives that

$$\mathrm{i}\{m_1\}v - v' = \mathrm{i}\alpha_2(\{m_1\} - \{m_2\})|m_2\rangle \in V.$$

Since $\{m_1\} \neq \{m_2\}$, then $|m_2\rangle \in V$ and hence $|m_1\rangle \in V$.

In order to prove that $V = \mathcal{D}$ we obtain from (19) and (20) that

$$\{R_{rs}(T_1) - \mathrm{i}sq^{m_2+1/2}R_{rs}(T_2)\}|m_2\rangle = \mathrm{i}rq^{1/2}|m_2 - 1\rangle$$
$$\{R_{rs}(T_1) + \mathrm{i}s^{-1}q^{-m_2+1/2}R_{rs}(T_2)\}|m_2\rangle = \mathrm{i}rq^{1/2}|m_2 + 1\rangle.$$

It follows from these relations that V contains the vectors $|m_2-1\rangle, |m_2-2\rangle, \ldots$ and the vectors $|m_2+1\rangle, |m_2+2\rangle, \ldots$. This means that $V = \mathcal{D}$ and the representation R_{rs} is irreducible. The proposition is proved. □

Note that the representations R_{rs} of proposition 2 turn into irreducible representations of the universal enveloping algebra $U(\mathrm{iso}_2)$ when $q \to 1$. For this reason, they are called *representations of the classical type*.

It is easy to show that the representations R_{rs} and $R_{r's'}$ of proposition 2 are equivalent if and only if $r' = \pm r$ and $s = q^m s$ for some $m \in \mathbb{Z}$.

Now let us show that if $r \in \mathbb{C}\backslash\{0\}$ and $s = \varepsilon\mathrm{i}q^{m+1/2}$, where $m \in \mathbb{Z}$ and $\varepsilon \in \{1, -1\}$, then the representation R_{rs} is reducible.

For this we note that the eigenvalues of the operator $R_{rs}(I)$ are

$$-\varepsilon\frac{q^n + q^{-n}}{q - q^{-1}} \qquad n = \pm\tfrac{1}{2}, \pm\tfrac{3}{2}, \pm\tfrac{5}{2}, \ldots$$

that is, every spectral point has multiplicity 2. The pairs of vectors $|-m+j\rangle$ and $|-m-j-1\rangle$, $j=0,1,2,\ldots$, are of the same eigenvalue. Let us define two subspaces V_1 and V_{-1} by the formulae $V_{\tilde{\varepsilon}} := \mathrm{lin}\{|j\rangle_{\tilde{\varepsilon}} \,|\, j=0,1,2,\ldots\}$, where

$$|j\rangle_{\tilde{\varepsilon}} := |-m+j\rangle + \tilde{\varepsilon}\mathrm{i}(-1)^j|-m-j-1\rangle \qquad j=0,1,2,\ldots. \tag{21}$$

A direct calculation shows that for $\tilde{\varepsilon}=1$ and for $\tilde{\varepsilon}=-1$ we have

$$R_{rs}(I)|j\rangle_{\tilde{\varepsilon}} = -\varepsilon\frac{q^{j+1/2}+q^{-j-1/2}}{q-q^{-1}}|j\rangle_{\tilde{\varepsilon}} \qquad j=0,1,2,\ldots \tag{22}$$

$$R_{rs}(T_2)|0\rangle_{\tilde{\varepsilon}} = -\frac{r}{q^{1/2}-q^{-1/2}}\left(\tilde{\varepsilon}|0\rangle_{\tilde{\varepsilon}} + \mathrm{i}|1\rangle_{\tilde{\varepsilon}}\right) \tag{23}$$

$$R_{rs}(T_2)|j\rangle_{\tilde{\varepsilon}} = -\varepsilon\frac{\mathrm{i}r}{q^{j+1/2}-q^{-j-1/2}}\left(|j+1\rangle_{\tilde{\varepsilon}} + |j-1\rangle_{\tilde{\varepsilon}}\right) \qquad j=1,2,3,\ldots \tag{24}$$

$$R_{rs}(T_1)|0\rangle_{\tilde{\varepsilon}} = \frac{r}{q^{1/2}-q^{-1/2}}\left(\tilde{\varepsilon}|0\rangle_{\tilde{\varepsilon}} + \mathrm{i}q|1\rangle_{\tilde{\varepsilon}}\right) \tag{25}$$

$$R_{rs}(T_1)|j\rangle_{\tilde{\varepsilon}} = \frac{\mathrm{i}r}{q^{j+1/2}-q^{-j-1/2}}\left(q^{j+1}|j+1\rangle_{\tilde{\varepsilon}} + q^{-j}|j-1\rangle_{\tilde{\varepsilon}}\right) \qquad j=1,2,3,\ldots. \tag{26}$$

These formulae show that the subspaces V_1 and V_{-1} are invariant with respect to the representation R_{rs}, that is, this representation is reducible.

Let us denote the restrictions of the considered above reducible representation R_{rs} to the invariant subspaces V_1 and V_{-1} by $R_r^{\varepsilon,1}$ and $R_r^{\varepsilon,-1}$, respectively. It is seen from formulae (22)–(26) that the operators are independent of s (this is a consequence of the equality $s=\varepsilon\mathrm{i}q^{m+1/2}$, $m\in\mathbb{Z}$, and the equivalence relations for the representations R_{rs}) and the index s is omitted. These formulae show that R_{rs}, $s=\varepsilon\mathrm{i}q^{m+1/2}$, is the direct sum of the representations $R_r^{\varepsilon,1}$ and $R_r^{\varepsilon,-1}$.

Using formulae (22)–(26) it is easy to prove that the representations $R_r^{\varepsilon,\tilde{\varepsilon}}$ and $R_{r'}^{\varepsilon',\tilde{\varepsilon}'}$ are equivalent if $(r,\varepsilon,\tilde{\varepsilon}) = (-r',\varepsilon',-\tilde{\varepsilon}')$.

Theorem 1. *Let r and r' be non-zero complex numbers such that $\mathrm{Re}\,r>0$ and $\mathrm{Re}\,r'>0$, and let $\varepsilon,\tilde{\varepsilon},\varepsilon',\tilde{\varepsilon}'\in\{1,-1\}$. If $(\varepsilon,\tilde{\varepsilon},r)\neq(\varepsilon',\tilde{\varepsilon}',r')$, then the representations $R_r^{\varepsilon,\tilde{\varepsilon}}$ and $R_{r'}^{\varepsilon',\tilde{\varepsilon}'}$ are irreducible and non-equivalent. Let $r\in\mathbb{C}\backslash\{0\}$, $\varepsilon,\tilde{\varepsilon}\in\{1,-1\}$, and let $r',s'\in\mathbb{C}\backslash\{0\}$, $s'\neq\pm\mathrm{i}q^{m+1/2}$ for all $m\in\mathbb{Z}$. Then the irreducible representations $R_r^{\varepsilon,\tilde{\varepsilon}}$ and $R_{r's'}$ are non-equivalent.*

Proof. The irreducibility is proved in the same way as in proposition 2. In order to prove a non-equivalence we note that the spectrum of the operator $R(I)$ for any of the representations $R_r^{+,+}$, $R_r^{+,-}$, $\mathrm{Re}\,r>0$, does not coincide with that of any of the representations $R_r^{-,+}$, $R_r^{-,-}$, $\mathrm{Re}\,r>0$. Therefore, any of the representations $R_r^{+,+}$, $R_r^{+,-}$, $\mathrm{Re}\,r>0$, cannot be equivalent to some of the representations $R_r^{-,+}$, $R_r^{-,-}$, $\mathrm{Re}\,r>0$.

The operators $R_r^{\varepsilon,\tilde{\varepsilon}}(T_2)$, $\varepsilon,\tilde{\varepsilon}=+,-$, are trace class operators. Their traces are non-zero (there exists only one non-zero diagonal matrix element with respect to the basis (21)). Since for $\mathrm{Re}\,r>0$ and $\mathrm{Re}\,r'>0$, $r\neq r'$, we have $\mathrm{Tr}\,R_r^{+,+}(T_2)\neq\mathrm{Tr}\,R_{r'}^{+,-}(T_2)$, then any of the representations $R_r^{+,+}$, $\mathrm{Re}\,r>0$, cannot be equivalent to some of the representations $R_r^{+,-}$, $\mathrm{Re}\,r>0$. It is proved similarly that any of the representations $R_r^{-,+}$, $\mathrm{Re}\,r>0$, cannot be equivalent to some of the representations $R_{r'}^{-,-}$, $\mathrm{Re}\,r'>0$. This proves the first part of the proposition. The second part is proved similarly. □

Representations $R_r^{\varepsilon,\tilde{\varepsilon}}$ from theorem 1 have no classical limit since at $q\to1$ the denominators in (22)–(26) turn into zero. For this reason, these representations are called *representations of non-classical type*. There are no analogues of such representations for the Lie algebra iso_2.

Theorem 2. *Every irreducible representation of $U_q(\mathrm{iso}_2)$ is equivalent to one of the representations R_{rs} and $R_{r'}^{\varepsilon,\tilde{\varepsilon}}$ or is one dimensional. This means that the representations R_{rs} and $R_{r'}^{\varepsilon,\tilde{\varepsilon}}$, $r, s, r' \in \mathbb{C}\backslash\{0\}$, $\varepsilon, \tilde{\varepsilon} \in \{1, -1\}$ defined by the relations (18)–(20) and (22)–(26), respectively, exhaust (up to equivalence) all irreducible representations of $U_q(\mathrm{iso}_2)$.*

Proof. Let R be an irreducible representation of $U_q(\mathrm{iso}_2)$. Then it follows from the definition of representations that $R(I)$ has some eigenvector $|0\rangle$. Thus there exists $s \in \mathbb{C}$, $s \neq 0$, such that

$$R(I)|0\rangle = \mathrm{i}[0]_{q,s}|0\rangle$$

where $[m]_{q,s} := (sq^m - s^{-1}q^{-m})/(q - q^{-1})$. Since R is irreducible there exists a complex number C such that $R(C_q) = C$ (see equation (6)). We define recursively the vectors

$$|j+1\rangle := R(\mathrm{i}T_1 - s^{-1}q^{-j+1/2}T_2)|j\rangle \qquad j = 0, 1, 2, \ldots \tag{27}$$

$$|j-1\rangle := R(\mathrm{i}T_1 + sq^{j+1/2}T_2)|j\rangle \qquad j = 0, -1, -2, \ldots. \tag{28}$$

Some of these vectors may be linearly dependent or equal to 0. It follows from (1)–(3) and (6) that

$$R(I)|j\rangle = \mathrm{i}[j]_{q,s}|j\rangle \qquad j \in \mathbb{Z} \tag{29}$$

$$R(\mathrm{i}T_1 + sq^{j+3/2}T_2)|j+1\rangle = -Cq|j\rangle \qquad j = 0, 1, 2, \ldots \tag{30}$$

$$R(\mathrm{i}T_1 - s^{-1}q^{-j+3/2}T_2)|j-1\rangle = -Cq|j\rangle \qquad j = 0, -1, -2, \ldots. \tag{31}$$

As a sample, we prove the relation (30) for $j \geqslant 0$:

$$\begin{aligned}
R(\mathrm{i}T_1 + sq^{j+3/2}T_2)|j+1\rangle &= R(\mathrm{i}T_1 + sq^{j+3/2}T_2)R(\mathrm{i}T_1 - s^{-1}q^{-j+1/2}T_2)|j\rangle \\
&= R(-T_1^2 + \mathrm{i}sq^{j+3/2}T_2T_1 - \mathrm{i}s^{-1}q^{-j+1/2}T_1T_2 - q^2T_2^2)|j\rangle \\
&= qR(-q^{-1}T_1^2 - qT_2^2 + \mathrm{i}sq^{j-3/2}T_1T_2 - \mathrm{i}s^{-1}q^{-j-1/2}T_1T_2)|j\rangle \\
&= qR(-q^{-1}T_1^2 - qT_2^2 - q^{-3/2}(1-q^2)T_1T_2I)|j\rangle \\
&= -qR(C_q)|j\rangle = -qC|j\rangle.
\end{aligned}$$

We obtain from (27) and (30) that

$$R(T_2)|j\rangle = -(s^{-1}q^{-j+1/2} + sq^{j+1/2})^{-1}(|j+1\rangle + Cq|j-1\rangle) \tag{32}$$

$$\mathrm{i}R(T_1)|j\rangle = \frac{sq^{j+1/2}}{s^{-1}q^{-j+1/2} + sq^{j+1/2}}|j+1\rangle + Cq\left(\frac{sq^{j+1/2}}{s^{-1}q^{-j+1/2} + sq^{j+1/2}}\right)|j-1\rangle. \tag{33}$$

Let us now consider two cases: (a) $C = 0$ and (b) $C \neq 0$.

(a) $C = 0$. Formulae (32) and (33) in this case give

$$R(T_2)|j\rangle = -(s^{-1}q^{-j+1/2} + sq^{j+1/2})^{-1}|j+1\rangle \tag{34}$$

$$\mathrm{i}R(T_1)|j\rangle = \frac{sq^{j+1/2}}{s^{-1}q^{-j+1/2} + sq^{j+1/2}}|j+1\rangle. \tag{35}$$

If the set $\{|j\rangle | j = 0, 1, 2, \ldots\}$ is linearly independent, it follows from (34) and (35) that $\mathrm{lin}\{|j\rangle, |j+1\rangle, \ldots\}$ is an invariant subspace for any $j = 1, 2, 3, \ldots$. Thus the representation is either reducible or one dimensional.

Now let there exist $l \in \mathbb{N}$ such that $|l\rangle$ is linearly dependent on linearly independent vectors $|0\rangle, |1\rangle, \ldots, |l-1\rangle$. Since the sequence of numbers $[j]_{q,s}$, $j \in \mathbb{Z}$, does not contain three equal elements, the only possible case is $|l\rangle = \alpha|k\rangle$ for some $k \in \{0, 1, \ldots, l-1\}$.

Let us consider the case $-sq^{l-1} \neq \pm\mathrm{i}$. For $\alpha \neq 0$ we find contradiction with the commutation relations (1)–(3) applying them to the vector $|l-1\rangle$. For $\alpha = 0$ and $l \geqslant 2$ we obtain the one-dimensional representation on the invariant subspace $\mathbb{C}|l-1\rangle$. For $\alpha = 0$ and $l = 1$ we must move attention to the vectors $|0\rangle, |-1\rangle, \ldots$ and the rest of the proof is fulfilled by repeating the above and below arguments except that we work with vectors with negative indices.

Now let us consider the case $-sq^{l-1} = \varepsilon\mathrm{i}$, $\varepsilon = \pm 1$. Since $sq^{l-1} = -s^{-1}q^{-l+1}$, the equalities (34) and (35) have no sense in this case.

For $\alpha \neq 0$ we find from equation $[l-1+j]_{q,s} = [l-1-j]_{q,s}$ (valid for all $j \in \mathbb{Z}$) that $k = l-2$. By (27), (30) and the relation $sq^{l-1} = -s^{-1}q^{-l+1}$ we have

$$|l\rangle = R(\mathrm{i}T_1 - s^{-1}q^{-l+3/2}T_2)|l-1\rangle = \alpha|k\rangle = \alpha R(\mathrm{i}T_1 + sq^{l-1/2}T_2)|l-1\rangle = 0.$$

This contradicts the equality $|l\rangle = \alpha|k\rangle$, $\alpha \neq 0$. For $\alpha = 0$ we have from (1) and from $R(\mathrm{i}T_1 - \varepsilon\mathrm{i}q^{1/2}T_2)|l-1\rangle = 0$ that

$$R(I)R(T_2)|l-1\rangle = \varepsilon\frac{q^2+1}{q^2-1}R(T_2)|l-1\rangle = \mathrm{i}[l]_{q,s}R(T_2)|l-1\rangle$$

and we can redefine $|l\rangle := R(T_2)|l-1\rangle$. From (3) and (30) we have $0 = R(\mathrm{i}T_1 + sq^{l+1/2}T_2)|l\rangle$.

If $|l\rangle$ is linearly dependent on $|0\rangle, |1\rangle, \ldots, |l-1\rangle$, there exists $\beta \in \mathbb{C}$ such that $|l\rangle = \beta|l-2\rangle$. As above, for $\beta \neq 0$ we find the contradiction $\beta|l-1\rangle = 0$ and for $\beta = 0$ we obtain a one-dimensional representation since $\mathbb{C}|l-1\rangle$ is an invariant subspace.

If $|l\rangle$ is linearly independent on the vectors $|0\rangle, |1\rangle, \ldots, |l-1\rangle$, we recursively redefine $|j+1\rangle := R(\mathrm{i}T_1 - s^{-1}q^{-j+1/2}T_2)|j\rangle$, $j = l, l+1, \ldots$. Then we again consider two cases.

- If there exists $\gamma \in \mathbb{C}$ such that $|l+l'\rangle = \gamma|l-2-l'\rangle$ for some $l' \in \{1, 2, \ldots, l-2\}$, then we obtain either a one-dimensional representation on the invariant subspace $\mathbb{C}|l+l'-1\rangle$ (when $\gamma = 0$) or a contradiction applying (1)–(3) to $|l-2-l'\rangle$.
- If $|l+l'\rangle$ is linearly independent on $|0\rangle, |1\rangle, \ldots, |l+l'-1\rangle$, then the representation is reducible since $\mathrm{lin}\{|j\rangle, |j+1\rangle, \ldots\}$ is invariant subspace for any j.

(b) $C \neq 0$. Consider first the case when $|l\rangle$ is linearly dependent on the linearly independent vectors $|0\rangle, |1\rangle, \ldots, |l-1\rangle$. It means that $|l\rangle = \alpha|k\rangle$ for some $k \in \{0, 1, \ldots, l-1\}$ and for $\alpha \in \mathbb{C}$. If $\alpha = 0$ we find a contradiction since

$$0 = |l\rangle = R(\mathrm{i}T_1 - s^{-1}q^{-l+3/2}T_2)|l-1\rangle = R(\mathrm{i}T_1 + sq^{l+1/2}T_2)|l\rangle = -Cq|l-1\rangle$$

implies $C = 0$. For $\alpha \neq 0$ we find a contradiction by applying (1)–(3) to the vector $|l-1\rangle$.

Now consider the case when the vectors $|j\rangle$, $j = 0, 1, 2, \ldots$, are linearly independent. If there exists $m \in \mathbb{N}$ such that the vector $|-m\rangle$ is linearly dependent on the linearly independent vectors $|j\rangle$, $j \in \{-m+1, -m+2, \ldots\}$, we write $|-m\rangle = \beta|p\rangle$ for some $\beta \in \mathbb{C}$ and $p > -m$. For $\beta = 0$ we find a contradiction similarly as in the above analogous cases. For $\beta = Cq$ and $p = -m+1$ we obtain the representation given (after some suitable rescaling of the basis) by (22)–(26). For $\beta = -Cq$ and $p = -m+2$ we can derive from (1) how operator $R(I)$ acts on the linearly independent vectors $|p\rangle$ and $R(T_2)|p-1\rangle$ and see that it cannot be diagonalized on this subspace. Therefore, this case is impossible. For other values of β and p we find a contradiction by applying (1)–(3) to $|-m+1\rangle$.

Thus the only possible remaining case is when all the vectors $|j\rangle$, $j \in \mathbb{Z}$ are linearly independent. Using the formulae (29), (32) and (33), in this case we obtain (after some suitable rescaling of the basis) the representation (18)–(20). The theorem is proved. □

It is clear from theorem 2 that for $q \in \mathbb{R}$ the irreducible $*$-representations of $U_q(\mathrm{iso}_2)$ which can be separated from representations of theorem 1, are equivalent to the irreducible

$*$-representations from [7]. However, it is not seen directly from formulae for representations since operators of representations in [7] are given with respect to a basis other than our one. Namely, the authors of [7] diagonalize the operator $R(T_2)$ which corresponds to the shifts in the group $ISO(2)$.

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